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(Nov–Dec)

FYUP : 3rd Semester Examination

MAJOR

MATHEMATICS

(Calculus–I and Statics)

MTH–200

Marks : 75

Time : 3 hours

The figures in the margin indicate full marks for the questions

Answer **one** question from each Unit

UNIT—I

1. *(a)* Show that a monotonic increasing sequence bounded above is convergent and it converges to its least upper bound. 5

(b) State whether the following statements are True or False with brief justification : 2×3=6

(i) A bounded sequence is always convergent.

(ii) The sequence $\left\{\frac{4n+3}{n+2}\right\}$ converges to 4.

(iii) If $\lim_{n\to\infty} u_n = 0$, then the series $\sum_{n=1}^{\infty} u_n$ is always convergent.

(c) State Cauchy's general principle of convergence of a sequence and apply it to show that the sequence $\{x_n\}$ is divergent if

$$x_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}$$ 1+4=5

(d) Is the sequence $\{n^2\}$ a Cauchy sequence? Justify your answer. 2

2. (a) Test the convergence of the following series (any *three*) : 2×3=6

(i) $\frac{1}{1.2.3} + \frac{1}{3.4.5} + \frac{1}{4.5.6} + \cdots \infty$

(ii) $\frac{1^2}{2} + \frac{2^2}{2^2} + \frac{3^2}{2^3} + \frac{4^2}{2^4} + \cdots$

(iii) $\sum_{n=1}^{\infty} \frac{1}{n^n}$

(iv) $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

(b) What is an alternating series? State Leibnitz's test for the convergence of an alternating series and show that the series $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$ converges. 1+1+4=6

(c) Show that the series

$$\sum \frac{\sqrt{n}x^n}{\sqrt{n^2+1}}, \quad x > 0$$

converges for $x < 1$ and diverges for $x \geq 1$. 4

(d) Test the absolute convergence of the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$. 2

UNIT—II

3. (a) Find the points on the curve $y = x^2 - 4x + 9$, the tangents at which pass through the origin. 4

(b) Find the radius of curvature at any point t on the curve

$$x = a(\cos t + t\sin t), \quad y = a(\sin t - t\cos t)$$ 3

(c) Find the asymptotes of

$$x^3+2x^2y+xy^2-x+1=0$$ 3

(d) Evaluate the following limits (any three) : 3×3=9

(i) $\underset{x\to 1}{\text{Lt}}\left\{\dfrac{x}{x-1}-\dfrac{1}{\log x}\right\}$

(ii) $\lim\limits_{x\to 0}\dfrac{\sin x - x}{x^3}$

(iii) $\lim\limits_{x\to 0}(\cos x)^{\frac{1}{x^2}}$

(iv) $\lim\limits_{x\to 1}\dfrac{\log(1-x)}{\cot(\pi x)}$

(v) $\underset{x\to\infty}{\text{Lt}}\dfrac{x^4}{e^x}$

4. (a) State and prove Taylor's theorem in Cauchy's form of remainder. 7

(b) Expand $\sin x$ in a finite series in power of x with remainder in Cauchy's form. 4

(c) Apply Maclaurin's theorem to $f(x)=(1+x)^4$ to deduce that

$$(1+x)^4=1+4x+6x^2+4x^3+x^4$$ 5

(d) Find the pedal equation of the curve $r=e^{\theta}$. 3

UNIT—III

5. (a) Two forces P and Q acting on a particle at an angle α have a resultant $(2k+1)\sqrt{P^2+Q^2}$. When they act at an angle $90°-\alpha$, the resultant becomes $(2k-1)\sqrt{P^2+Q^2}$. Prove that

$$\tan\alpha=\frac{k-1}{k+1}$$ 5

(b) State and prove Lami's theorem. 1+4=5

(c) If the two like-parallel forces P and Q acting on a rigid body at A and B be interchanged in position, then show that the point of application of the resultant will be displaced along $\overrightarrow{AB}$ through a distance d where

$$d=\frac{P-Q}{P+Q}AB \quad (P>Q)$$ 5

(d) Three forces P, Q, R acting at the vertices A, B, C respectively of a triangle, each perpendicular to the opposite side, keep it in equilibrium. Prove that $P:Q:R=a:b:c$. 4

6. (a) Three forces acting along the sides of a triangle, taken in order, are equivalent to a couple. Show that their magnitudes are proportional to the sides of the triangle. 5

(b) Three forces P, Q, R act in the same sense along the sides $\overrightarrow{BC}$, $\overrightarrow{CA}$, $\overrightarrow{AB}$ of a triangle ABC. Show that, if their resultant passes through the in-centre, then $P+Q+R=0$. 4

(c) If the resultant of two equal forces inclined at an angle 2θ is twice as great as when they are inclined at an angle 2ϕ, then prove that $\cos\theta = 2\cos\phi$. 5

(d) Forces of magnitudes 1, 2, 3, 4, $2\sqrt{2}$ act respectively along the sides $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{CD}$, $\overrightarrow{DA}$ and the diagonal $\overrightarrow{AC}$ of the square $ABCD$. Show that their resultant is a couple, and find its moment. 5

UNIT—IV

7. (a) Forces proportional to 1, 2, 3, 4 act along the sides $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{AD}$, $\overrightarrow{DC}$ respectively of a square $ABCD$, the length of whose sides is 2 m. Find the magnitude and line of action of the resultant. 5

(b) A heavy uniform rod of length $2a$ rests in equilibrium, having one end against a smooth vertical wall, and being placed upon a peg at a distance b from the wall. Show that the inclination of the rod to the horizontal is $\cos^{-1}\left(\frac{b}{a}\right)^{\frac{1}{3}}$. 5

(c) A uniform ladder is in equilibrium with one end resting on the ground and the other against a vertical wall; if the ground and wall be both rough, the coefficients of friction being μ and μ' respectively, and if the ladder be on the point of slipping on both ends, then show that the inclination of the ladder to the horizon is given by $\tan\theta = \frac{1-\mu\mu'}{2\mu}$. 5

(d) Find the centre of gravity of a parallelogram formed by four uniform rods. 4

8. (a) The moments of a system of coplanar forces acting in the (x, y)-plane about $(0, 0)$, $(a, 0)$, $(0, a)$ are aw, $2aw$, $3aw$ respectively. Find the magnitudes of the components parallel to the coordinate axes and the line of action of the single force to which the system is equivalent. 5

(b) The least force which will move a weight up an inclined plane is P. Show that the least force, acting parallel to the plane, which will move the weight upwards is $P\sqrt{1+\mu^2}$, where μ is the coefficient of friction. 5

(c) Forces P, Q, R, S act along the sides $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{CD}$, $\overrightarrow{DA}$ of the cyclic quadrilateral $ABCD$, taken in order, where A and B are the extremities of a diameter. If they are in equilibrium, then prove that

$$R^2 = P^2 + Q^2 + S^2 + \frac{2PQS}{R}$$

5

(d) A uniform rod AB is suspended with its end in contact with a smooth vertical wall $\overrightarrow{AC}$ by a string $\overrightarrow{CE}$. Show that CB is horizontal if $AE = \frac{1}{3}AB$. 4

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